Discrete Darboux transformation for discrete polynomials of hypergeometric type

1998 J. Phys. A: Math. Gen. 31 2191

(http://iopscience.iop.org/0305-4470/31/9/009)

J. Phys. A: Math. Gen. **31** (1998) 2191–2196. Printed in the UK PII: S0305-4470(98)87831-1

Discrete Darboux transformation for discrete polynomials of hypergeometric type

Gaspard Bangerezako†‡

Institut de Mathématique, Université Catholique de Louvain, Chemin du Cyclotron, 2, B-1348 Louvain-La-Neuve, Belgium

Received 25 September 1997, in final form 18 December 1997

Abstract. The Darboux transformation, well known in second-order differential operator theory, is applied to the difference equations satisfied by the discrete hypergeometric polynomials (Charlier, Meixner-Kravchuk, Hahn).

1. Introduction

Since Darboux, who showed how $z(x) = Ay(x) + By'(x)$ solves $z''(x) = (\xi(x) + h)z(x)$ when $y(x)$ satisfies $y''(x) = (\eta(x) + h)y(x)$ [3], numerous generalizations have been investigated.

Consider first the second-order difference equation

$$H(x; j)\Omega(x; j) = 0 \tag{1}$$

where

$$H(x; j) = E^2 + v(x; j)E + u(x; j) \tag{2}$$

with

$$E^i\Omega(x; j) = \Omega(x + i; j) \tag{3}$$

$x \in R$, $i, j \in Z$.

Suppose that one can form the products

$$\begin{aligned} H(x; j) - \mu(j) &= (E + g(x; j))(E + f(x; j)) \\ H(x; j + 1) - \mu(j) &= (E + f(x; j))(E + g(x; j)) + \alpha(j) \end{aligned} \tag{4}$$

then, the operator $H(x; j + 1)$ is called a *discrete Darboux transformation* of $H(x; j)$. $E + g(x; j)$ and $E + f(x; j)$ are said to play the role of '*lowering*' and '*raising*' operators respectively. From (4), we have the following commutation relation

$$H(x; j + 1)(E + f(x; j)) = (E + f(x; j))(H(x; j) + \alpha(j)) \tag{5}$$

which is a discrete analogue of the so-called *dressing chain* [12, 14]. The dressing chain (5) is equivalent to the system

$$\begin{aligned} f(x; j) + g(x + 1; j) &= f(x + 1; j + 1) + g(x; j + 1) \\ f(x; j)g(x; j) + \alpha(j) &= f(x; j + 1)g(x; j + 1) + \mu(j + 1) - \mu(j). \end{aligned} \tag{6}$$

† Permanent address: Université du Burundi, Faculté des sciences, Département de Mathématique, BP 2700 Bujumbura, Burundi, East Central Africa.

‡ E-mail address: bangerezako@agel.ucl.ac.be

0305-4470/98/092191+06$19.50

In the continuous case, many questions concerning the intrinsic structure (Hamiltonian, integrability, etc) of such chains are explained in [12, 11]. In the discrete case, similar structures remain generally obscure. Even some particular considerations of such systems appearing today are mainly directed to the cases in which the shift operator in (3) acts on j but not on x. This is typically the case when one is treating the discrete Schrödinger problem or particularly the polynomial recurrence relations. Note that except for some particular cases of self-adjointness of x and j, this nuance is very significant. The polynomial recurrence relations case for example, is characterized by the linearity of the eigenvalue, which is then x, which *a priori* facilitates the application of the Darboux transformation techniques.

In this work, our discussions will be confined to a restricted case of the discrete Darboux transformation (4). Namely, it will be presumed that j acts not simply as a symbol (index) as in [12, 11], but as an independent variable. In that situation, one says that the operator H is *factorizable* according to the Infeld–Hull method [9, 4]. For convenience, we will call j the variable of factorization.

Let us note that when dealing with the second-order hypergeometric difference operator on a linear lattice [10], one can adapt the Infeld–Hull factorization so that the latter one becomes equivalent to the Nikiforov–Suslov–Uvarov theory [10] as was shown in [13]. As noted in [13], the cited equivalence remains valid when one passes from a linear to nonlinear lattice. In the latter case, the role of the 'lowering' operator is played by the Askey–Wilson derivative [1, 6, 10]. A similar factorization was used (implicitly) in [5] to give a very simplified version of the proof of the orthogonality relation for the Askey–Wilson polynomials [1]. Next, in [15], it was proven that, starting from the recurrence relations for the Tchebyshev polynomials, one can obtain, using the factorization procedure, the corresponding relations for some special cases of the Askey–Wilson polynomials. It follows from this observation that [17] those special cases of Askey–Wilson polynomials are (not only discrete classical) continuous semiclassical polynomials [8].

Here, as in [13], we are dealing with the factorization of the second-order difference operator on a linear lattice,

$$\sigma(x)\Delta\nabla + \tau(x)\Delta - \lambda \tag{7}$$

where $\Delta = E - 1$, $\nabla = 1 - E^{-1}$ (see (3) for the definition of E^i), σ and τ being polynomials of degree $\leqslant 2$ and 1 respectively, λ being a constant (in x). However, the particularity of this work resides in that the 'variable of factorization' (as j in (4)) is exactly the degree n of the corresponding polynomials. So that during the procedure of transformation, only the term λ is altered. We will see that this phenomenon is characteristic of the discrete hypergeometric polynomials on a linear lattice. In the next section, we shall give and discuss the announced factorization of the operator (7). In the last section, we shall apply the result of the factorization to the classical orthogonal polynomials of a discrete variable on a linear lattice (Charlier, Meixner-Kravchuk, Hahn). Similar factorizations having been obtained (differently) for the Charlier and Meixner-Kravchuk cases in [9], we first succeed to handle the Hahn case, specialized by the nonlinearity of the eigenvalue, as a function of j.

2. Finite difference analogues of $\lambda(n)$-eigenfunctions of hypergeometric type

Let $\Phi(x; n)$, $x \in R$, $n \in Z$, be a given system of hypergeometric functions such that

$$(\sigma(x)\Delta\nabla + \tau(x)\Delta)\Phi(x; n) = \lambda(n)\Phi(x; n) \tag{8}$$

where $\sigma(x)$ and $\tau(x)$ are polynomials of degree less than or equal to 2 and 1 respectively, $\lambda(n) = n\tau' + \frac{1}{2}n(n-1)\sigma''$.

For convenience, we shall call them $\lambda(n)$-*eigenfunctions*. It is clear that this set includes *the discrete polynomials of hypergeometric type* [10]. Let us note that in this case the number n is the degree of the corresponding polynomials.

One can easily find some function $\rho(x)$ such that (8) is equivalent to

$$(E^2 - [2\sigma(x+1) + \tau(x+1) + \lambda(n)]E + (\sigma(x) + \tau(x))\sigma(x+1))(\rho(x)\Phi(x;n)) = 0 \tag{9}$$

with

$$\frac{\rho(x+1)}{\rho(x)} = \sigma(x) + \tau(x).$$

Let $L = E^2 - [2\sigma(x+1) + \tau(x+1)]E + (\sigma(x) + \tau(x))\sigma(x+1)$ and

$$H(x;n) = L - \lambda(n)E = E^2 - [2\sigma(x+1) + \tau(x+1) + \lambda(n)]E \\ +(\sigma(x) + \tau(x))\sigma(x+1).$$

Supposing the existence of two polynomials $f(x;n)$ and $g(x;n)$ of second degree with identical leading coefficients such that

$$H(x;n) - \mu(n) = (E + g(x;n))(E + f(x;n)) \tag{10}$$

for some constant $\mu(n)$, one can verify that

$$(E + f(x;n))(E + g(x;n)) = H(x;n') - \mu(n) \tag{11}$$

where

$$\lambda(n') = \lambda(n) + \Delta(f(x;n) - g(x;n)) \tag{12}$$

n' being some function of n, which will be determined later.

Equations (10) and (11) give

$$H(x;n')(E + f(x;n)) = (E + f(x;n))H(x;n). \tag{13}$$

In order to determine $f(x;n)$ and $g(x,n)$, one needs to note that equation (10) leads to the system

$$\begin{aligned} f(x+1;n) + g(x;n) &= -2\sigma(x+1) - \tau(x+1) - \lambda(n) \\ f(x;n)g(x;n) &= (\sigma(x) + \tau(x))\sigma(x+1) - \mu(n) \end{aligned} \tag{14}$$

which is in fact a discrete Riccati equation.

Setting

$$\begin{aligned} f(x;n) &= -\sigma(x) - \tau(x) - \tfrac{1}{2}\lambda(n) + \varphi(x;n) \\ g(x;n) &= -\sigma(x+1) - \tfrac{1}{2}\lambda(n) - \varphi(x+1;n) \end{aligned} \tag{15}$$

the first equation in (14) will automatically be verified. The second reads

$$\tfrac{1}{2}\lambda(n)(\sigma(x+1) + \sigma(x) + \tau(x)) + \tfrac{1}{4}\lambda^2(n) + \mu(n) \\ +(\sigma(x) + \tau(x))\varphi(x+1;n) - \sigma(x+1)\varphi(x;n) \\ +\tfrac{1}{2}\lambda(n)\Delta\varphi(x;n) - \varphi(x;n)\varphi(x+1;n) = 0 \tag{16}$$

a discrete Riccati equation related to $\varphi(x;n)$. Looking for polynomial solutions of degree $\leqslant 1$, $\varphi(x;n) = \phi(n)x + \psi(n)$; knowing that $\sigma(x) = \sigma_0 x^2 + \sigma_1 x + \sigma_2$, $\tau(x) = \tau_0 x + \tau_1$

and equating coefficients on the left-hand side of (16) to zero, one finds two possible sets of solutions

$$\phi_1(n) = \tau_0 + (n-1)\sigma_0 \qquad \phi_2(n) = -n\sigma_0 \tag{17}$$

$$\psi_{1,2}(n) = \frac{\phi_{1,2}(n)(\tau_1+\tau_0-\sigma_0-\phi_{1,2}(n)) + \lambda(n)\sigma_0 + \lambda(n)\sigma_1 + \frac{1}{2}\lambda(n)\tau_0}{2\phi_{1,2}(n)+2\sigma_0-\tau_0} \tag{18}$$

$$\mu_{1,2}(n) = \psi_{1,2}(n)(\psi_{1,2}(n)+\phi_{1,2}(n)+\sigma_1+\sigma_0-\tau_1) - \tfrac{1}{2}\lambda(n)(\sigma_0+\sigma_1+2\sigma_2+\tau_1)$$
$$-\phi_{1,2}(n)(\sigma_2+\tau_1+\tfrac{1}{2}\lambda(n)) - \tfrac{1}{4}\lambda^2(n). \tag{19}$$

On the other side, (15) reads

$$f(x;n) = -\sigma_0 x^2 + (\phi(n)-\sigma_1-\tau_0)x + \psi(n) - \sigma_2 - \tau_1 - \tfrac{1}{2}\lambda(n) \tag{20}$$

$$g(x;n) = -\sigma_0 x^2 - (\phi(n)+2\sigma_0+\sigma_1)x - \sigma_0 - \sigma_1 - \sigma_2 - \tfrac{1}{2}\lambda(n) - \phi(n) - \psi(n). \tag{21}$$

Thus, the conditions advanced in (10) are all satisfied.

Next, from (20) and (21), we obtain $\Delta(f-g) = 2\phi(n) + 2\sigma_0 - \tau_0$, and using (17), it follows $(\lambda(n) = n\tau_0 + n(n-1)\sigma_0)$,

$$(\Delta(f-g))_1 = 2\phi_1(n) + 2\sigma_0 - \tau_0 = \tau_0 + 2n\sigma_0 = \lambda(n+1) - \lambda(n) \tag{22}$$

$$(\Delta(f-g))_2 = 2\phi_2(n) + 2\sigma_0 - \tau_0 = -(\tau_0 + 2(n-1)\sigma_0) = \lambda(n-1) - \lambda(n). \tag{23}$$

Referring to (12), this means that we have proved that $n'_{1,2} = n \pm 1$ and (13) reads

$$H(x;n\pm 1)(E + f_{1,2}(x;n)) = (E + f_{1,2}(x;n))H(x;n) \tag{24}$$

which is the searched commutation relation (5) ($j := n$).

From (24) and (9), it obviously follows that for any $\lambda(n)$-eigenfunction $\Phi(x;n)$ of hypergeometric type, the following difference relations are valid

$$c_1(n)\tilde{\Phi}(x;n+1) = (E + f_1(x;n))\tilde{\Phi}(x;n) \tag{25}$$

$$c_2(n)\tilde{\Phi}(x;n-1) = (E + f_2(x;n))\tilde{\Phi}(x;n) \tag{26}$$

where $\tilde{\Phi}(x;n) = \rho(x)\Phi(x;n)$,

$$f_{1,2}(x;n) = -\sigma_0 x^2 + (\phi_{1,2}(n) - \sigma_1 - \tau_0)x + \psi_{1,2}(n) - \sigma_2 - \tau_1 - \tfrac{1}{2}\lambda(n).$$

From this, of course, the recurrence relations, for $\Phi(x;n)$, can be deduced. Moreover we see that the 'raising' operator in (25) leads to the Rodrigues-type formula. One now needs to remark from (16) and (22) that, conversely, the possibility of such a factorization on a type (7) operator implies necessarily that $\lambda^z(n) = \frac{1}{2}\sigma'' n^2 + ((\tau' - \frac{1}{2}\sigma'')^2 + 2\sigma'' z)^{\frac{1}{2}} n + z$. Whence, operator (7) is factorizable (with $j := n$) iff the corresponding polynomials are the discrete hypergeometric polynomials on a linear lattice (Charlier, Meixner-Kravchuk and Hahn cases corresponding to $z = 0$) or their trivial generalizations. It can be checked [2] that this characteristic property can be extended not only to all classical (Askey–Wilson) polynomials but also to the discrete semiclassical ones [6, 7], so to include related properties obtained in [16].

3. Examples

Consider now the equation,

$$\tilde{L}(x;n)Y(x;n) = 0 \tag{27}$$

Table 1. Data for the Charlier case.

$H(x;n)$	$E^2-(x+\mu+\lambda(n)+1)E+\mu(x+1)$
$\rho(x)$	μ^x
$f_1(x;n)$	$-x+n$
$f_2(x;n)$	$-\mu$
$g_1(x;n)$	$-\mu$
$g_2(x;n)$	$-x+n-1$
$\mu_1(n)$	$\mu n+\mu$
$\mu_2(n)$	μn

Table 2. Data for the Meixner case.

$H(x;n)$	$E^2-[(\mu+1)x+\mu(\gamma+1)+1+\lambda(n)]E+\mu x^2+\mu(\gamma+1)x+\gamma\mu$
$\rho(x)$	$\mu^x\Gamma(x+\gamma)$
$f_1(x;n)$	$-x+n$
$f_2(x;n)$	$-\mu(x+\gamma+n)$
$g_1(x;n)$	$-\mu(x+\gamma+n+1)$
$g_2(x;n)$	$-x+n-1$
$\mu_1(n)$	$\mu(n\gamma+n^2+n+\gamma)$
$\mu_2(n)$	$\mu n(\gamma+n-1)$

Table 3. Data for the Hahn case.

$H(x;n)$	$E^2+[2x^2+(6+\beta-\alpha-2N)x+(5+2\beta-\alpha-3N-\beta N$
	$-\lambda(n))]E+[x^4+(4+\beta-\alpha-2N)x^3+(6+3\beta-3\alpha-6N$
	$+N^2-2N\beta+\alpha N-\alpha\beta)x^2+(4+3\beta-3\alpha-6N+2N^2$
	$-4N\beta+2N\alpha-2\alpha\beta+N^2\beta+N\alpha\beta)x+1+\beta-\alpha-2N$
	$+N^2-2N\beta+\alpha N-\alpha\beta+N^2\beta+N\alpha\beta$
$\rho(x)$	$\frac{\Gamma(x+\beta+1)}{\Gamma(-x+N)}$
$f_1(x;n)$	$x^2-(N+\alpha+n-1)x-(\beta+1)(N-1)-\frac{1}{2}\lambda(n)+\psi_1(n)$
$g_1(x;n)$	$x^2+(3+n+\beta-N)x+2+\beta+n-N-\frac{1}{2}\lambda(n)-\psi_1(n)$
$\mu_1(n)$	$\psi_1(n)(\psi_1(n)-1-\beta N-n)-\frac{1}{2}\lambda(n)(\beta+1)(N-1)$
	$-\frac{1}{2}\lambda(n)(N+\alpha-1)+(n+\alpha+\beta+1)(\beta+1)(N-1)$
	$+\frac{1}{4}\lambda(n)(n+2)(n+\alpha+\beta+1)$
$f_2(x;n)$	$x^2+(2+\beta-N+n)x-(\beta+1)(N-1)-\frac{1}{2}\lambda(n)+\psi_2(n)$
$g_2(x;n)$	$x^2+(2-n-N-\alpha)x-N-\alpha-n+1-\frac{1}{2}\lambda(n)-\psi_2(n)$
$\mu_2(n)$	$\psi_2(n)(\psi_2(n)+n+\alpha-\beta N+\beta)-\frac{1}{2}\lambda(n)(N+\alpha-1)$
	$-n(\beta+1)(N-1)-\frac{1}{2}\lambda(n)(\beta+1)(N-1)-\frac{1}{2}\lambda(n)n-\frac{1}{4}\lambda^2(n)$
$\psi_1(n)$	$\frac{(n+\alpha+\beta+1)(\beta+1)(N-1)-\lambda(n)(N+\alpha)+\frac{1}{2}\lambda(n)(\alpha+\beta+2)}{2+2n+\alpha+\beta}$
$\psi_2(n)$	$\frac{n(\beta+1)(N-1)+\lambda(n)(N+\alpha)-\frac{1}{2}\lambda(n)(\alpha+\beta+2)}{2n+\alpha+\beta}$

where $\tilde{L}$ is the operator given in (7). Here, we define the discrete classical polynomials on a linear lattice as the non-trivial polynomial solutions of (27). From (25), it is clear that if $P(x;n)$ is such a solution, then

$$P(x;n)=\frac{c(n)}{\rho(x)}\prod_{i=0}^{n-1}(E+f(x;i))\rho(x) \tag{28}$$

$c(n)$ being some constant (in x). Next, we can indentify them according to the corresponding choices of σ and τ [10]. The Charlier polynomials correspond to $\sigma(x) = x$ and $\tau(x) = \mu - x$. For the Meixner and Hahn polynomials we have respectively $\sigma(x) = x$; $\tau(x) = \gamma\mu - x(1-\mu)$ and $\sigma(x) = x(N+\alpha-x)$; $\tau(x) = (\beta+1)(N-1)-(\alpha+\beta+2)x$. In [10] one can find explicit formulae for their coefficients (from corresponding 'hypergeometric series') but we are not concerned with those here.

Direct substitutions in the expressions obtained in section 2 lead to the necessary data for the factorization of the Charlier (table 1), Meixner (table 2) and Hahn (table 3) cases.

It is clear that the same technique can also be applied to the q-versions of the preceding polynomials. Extension to other difference operators is in progress [2].

Acknowledgments

We would like to thank Professor A P Magnus for suggesting the present explorations and for fruitful discussions. Thanks are also addressed to the Belgian General Agency for Cooperation with Developing Countries (AGCD) for financial support.

References

[1] Askey R and Wilson J 1985 Some basic hypergeometric orthogonal polynomials that generalize Jacobi polynomials *Mem. Am. Math. Soc.* **54** 1–55
[2] Bangerezako G and Magnus A P The factorization method and the semi-classical orthogonal polynomials, in preparation
[3] Darboux G 1915 *Théorie des Surfaces* vol II (Paris: Gauthier-Villars) p 213
[4] Infeld L and Hull T 1951 The factorization method *Rev. Mod. Phys.* **23** 21–68
[5] Kalnins E G and Miller W 1989 Symmetry techniques for q-series: Askey–Wilson polynomials *Rocky Mountain J. Math.* **19** 223–30
[6] Magnus A P 1988 *Associated Askey–Wilson Polynomials as Laguerre–Hahn Orthogonal Polynomials (Lecture Notes in Mathematics 1329)* (Berlin: Springer) pp 261–78
[7] Magnus A P 1997 Special non uniform lattice(snul) orthogonal polynomials on discrete dense sets of points *J. Comput. Appl. Math.* **65** 253–65
[8] Magnus A P 1984 *Riccati Acceleration of Jacobi Continued Fractions and Laguerre–Hanh Orthogonal Polynomials (Lecture Notes in Mathematics 1071)* (Berlin: Springer) pp 213–30
[9] Miller W Jr 1969 Lie theory and difference equations I *J. Math. Anal. Appl.* **28** 383–99
[10] Nikiforov A, Suslov S and Uvarov V 1991 *Classical Orthogonal Polynomials of a Discrete Variable (Springer Series in Computational Physics)* (Berlin: Springer)
[11] Shabat A 1992 The infinite dimensional dressing dynamical system *Inverse Problems* **8** 303–8
[12] Shabat A B and Veselov A P 1993 Dressing chain and spectral theory of Schrödinger operator *Funct. Anal. Appl.* **27** 81–96 (translation from 1993 *Funk. Anal. Pril.* **27** 1–21)
[13] Smirnov Yu F 1996 On factorization and algebraization of difference equations of hypergeometric type *Proc. Int. Workshop on Orthogonal Polynomials in Mathematical Physics (Leganés, 24–26 June)* ed M Alfaro *et al*
[14] Spiridonov V, Vinet L and Zhedanov A 1993 Difference Schrödinger operators with linear and exponential discrete spectra *Lett. Math. Phys.* **29** 63–73
[15] Spiridonov V, Vinet L and Zhedanov A 1995 Discrete Darboux transformations, the discrete-time Toda lattice and the Askey–Wilson polynomials *Meth. Appl. Anal.* **2** 369–98
[16] Spiridonov V, Vinet L and Zhedanov A 1997 Spectral transformations, self-similar reductions and orthogonal polynomials *J. Phys. A: Math. Gen.* **30** 7621–37
[17] Zhedanov A 1997 Rational spectral transformations and orthogonal polynomials *J. Comput. Appl. Math.* **85** 67–86